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Complemented lattice-ordered groups

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ABSTRACT

This article introduces complemented lattice-ordered groups and studies how they interact with the space of minimal prime convex ℓ-subgroups. Along the way the concept of rigidity turns out to be of paramount importance. Another important tool: the notion of a z-subgroup, introduced some time ago, but, until now, left largely to languish.

1. INTRODUCTION

We begin by reviewing some pertinent information from the general theory of lattice-ordered groups. Our standard reference is [BKW]. First of all, we shall employ the common abbreviations: ℓ-group for "lattice-ordered group; ℓ-subgroup for a subgroup which is at once a sublattice, etc. A convex ℓ-subgroup is an ℓ-subgroup which is (order) convex. If G is any ℓ-group, $C(G)$ will stand for the lattice of all convex ℓ-subgroups of G. It is well known – due to G. Birkhoff – that $C(G)$ is a distributive lattice; hence it is also Brouwerian, which means that in $C(G)$ finite intersections distribute over arbitrary suprema.

A convex ℓ-subgroup of G which is also normal in G is called an ℓ-ideal.

If G is any ℓ-group and $X \subseteq G$, then $X' = \{g \in G: |g| \wedge |x| = 0\}$. X' is called the *polar* of X, and it is a convex ℓ-subgroup of G. We say (without qualification) that $T \in C(G)$ is a *polar of* G if $T = T''$. The set $P(G)$ of all polars of G

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is a boolean algebra under inclusion, a fact first proved by Šik [Š]. Recall that if $x \in G$ then $G(x)$ denotes the convex ℓ-subgroup generated by x. Note that $G(|x|)=G(x)$, and that $(G(x))'=\{x\}'$; we denote the latter by x', and then expect the meaning of the notation x'' be clear. The subset $\mathbf{Pr}(G)=\{x'' \colon x \in G\}$ is a sublattice of $\boldsymbol{P}(G)$, in view of the identities $(x \wedge y)''=x'' \cap y''$ and $x'' \bigsqcup y''= =(x \vee y)''$, valid for all $0 \leq x, y \in G$; (note: the symbol $\bigsqcup$ denotes the supremum in $\boldsymbol{P}(G)$; in general, $\boldsymbol{P}(G)$ is not a sublattice of $\boldsymbol{C}(G)$.)

We call $\mathbf{Pr}(G)$ the *lattice of principal polars.*

If A and B are convex ℓ-subgroups of G and $G=A+B$, while $A \cap B=0$, then we say that G *is the cardinal sum of A and B*, and write $G=A \boxplus B$. If this should be the case then it is clear that both A and B are polars and that $A'=B$. If $G=A \boxplus B$ then we say that A and B are *cardinal summands* of G. In view of the distributivity of $\boldsymbol{C}(G)$, it should be obvious that $\boldsymbol{S}(G)$, the family of cardinal summands of G, is both a sublattice of $\boldsymbol{C}(G)$ and a *subalgebra* of $\boldsymbol{P}(G)$.

A convex ℓ-subgroup P of the ℓ-group G is said to be *prime* if $a \wedge b=0$ implies that either $a \in P$ or $b \in P$. There are many conditions which can be used to characterize prime subgroups - see [BKW], Theorem 2.4.1. - but we shall limit ourselves to mentioning that P is prime in $\boldsymbol{C}(G)$ if and only if the set G/P of right cosets is a totally ordered set under the natural factor ordering. If P is an ℓ-ideal then P is prime precisely when G/P is an o-group. (Recall: o-group is the common abbreviation for a totally ordered group.)

From the above paragraph one can easily obtain that the set of all primes, $\mathbf{Spec}(G)$, forms a *root system* under inclusion; that is to say, no two incomparable elements exceed a third. By Zorn's Lemma, every prime contains a minimal prime - see [BKW], 2.4.5. - and we also have the following criterion which distinguishes in $\mathbf{Spec}(G)$ the minimal ones: ([BKW], Theorem 3.4.13.)

1.0. *$P \in \mathbf{Spec}(G)$ is minimal exactly when $a \in P$ implies that $a' \nsubseteq P$. Alternately: P is minimal if and only if $a'' \boxplus a' \nsubseteq P$, for each $a \in G$.*

This fact is a consequence of a more fundamental result which connects minimal primes and ultrafilters of the positive cone of an ℓ-group. There are many contributors in the literature: see [B], [JK], [By] and [CMc]; it also appears in [BKW] as Theorem 3.4.10. We shall refer to it as *the Lemma on Ultrafilters*.

In an ℓ-group G a *filter* is a subset F of positive elements excluding 0, so that $a \wedge b \in F$ whenever a and b are in F. A maximal filter is called an *ultrafilter*.

LEMMA ON ULTRAFILTERS. *Let G be an ℓ-group. For each minimal prime P, the set $u(P)=\{q>0 \colon q \notin P\}$ is an ultrafilter. Conversely, if U is an ultrafilter then $Q(U)=\bigcup\{x' \colon x \in U\}$ is a minimal prime. The correspondences $P \rightarrow u(P)$ and $U \rightarrow Q(U)$ are mutually inverse bijections.*

$\mathbf{Min}(G)$ stands for the subset of $\mathbf{Spec}(G)$ consisting of all the minimal primes of G. We proceed to endow $\mathbf{Spec}(G)$ with a topology (and $\mathbf{Min}(G)$ with the subspace topology.) This is none other than the "hull-kernel" topology, in which the basic open sets are, for $\mathbf{Spec}(G)$, and for $a \in G$:

$$X(a) = \{P \in \mathbf{Spec}(G)\colon a \notin P\},$$

and for $\mathbf{Min}(G)$:

$$M(a) = \{P \in \mathbf{Min}(G)\colon a \notin P\}.$$

Here are some elementary observations; the proofs are straightforward. In [LZ] $\mathbf{Spec}(G)$ and $\mathbf{Min}(G)$ are thoroughly discussed for vector lattices; those authors, in turn, point to the earlier work of Amemiya [Am] and Johnson and Kist [JK].

1.1. $\mathbf{Min}(G)$ *has the* T_3 *separation property*: *any point and any closed set excluding that point can be separated by disjoint open sets. In particular,* $\mathbf{Min}(G)$ *is Hausdorff.*

1.2. *In general,* $\mathbf{Spec}(G)$ *is not Hausdorff. Indeed, this occurs exactly when every prime of G is maximal. The ℓ-groups which satisfy this condition are the hyper-archimedean ℓ-groups.* (These ℓ-groups are reasonably well understood. They were investigated by Conrad in [C1] and by Martinez in [M]. From [C1] we record two conditions which are equivalent to G being hyper-archimedean:
1.2. a) Every principal convex ℓ-subgroup of G is a cardinal summand.
1.2. b) G admits a representation as an ℓ-subgroup of the ℓ-group of real-valued functions R^I on the set I, so that if $0<a$, $b \in G$ then there is a natural number n for which $na(i) \geq b(i)$, for each $i \in I$ which satisfies $a(i)>0$. Moreover, if this is the case then every representation of G as real-valued functions satisfies this property.)

1.3. $\mathbf{Spec}(G)$ *is compact if and only if G has a strong unit.* (Recall: $u>0$ in G is called a *strong-unit* if $G(u)=G$. By contrast, we speak of a *unit* (or *weak unit*) $u \in G$, if $u'=0$. Clearly, a strong unit is a unit.)

1.4. *For each* $P \in \mathbf{Spec}(G)$, *the closure* $cl\{P\} \supseteq \{Q \in \mathbf{Spec}(G)\colon Q \supseteq P\}$. *Therefore if* $\{P\}$ *is closed the prime P is maximal. We can then deduce* 1.2 *and more*; *for an ℓ-group G the following are equivalent*:

a) *G is hyper-archimedean.*
b) $\mathbf{Spec}(G)$ *is Hausdorff.*
c) $\mathbf{Spec}(G)$ *has the* T_1 *separation property.*
d) $\mathbf{Spec}(G) = \mathbf{Min}(G)$.

For any ℓ-group whatsoever, $cl(\mathbf{Min}(G)) = \mathbf{Spec}(G)$.

Beginning with the next section and for the rest of the article we shall concentrate our attention on $\mathbf{Min}(G)$ and omit any mention of the full spectrum of primes.

2. RIGID SUBGROUPS

To begin, let us recall some terminology from [CM2]: a convex ℓ-subgroup C of G is said to be *very large* in G if no minimal prime contains C. For example, $a''\boxplus a'$ is very large in G, for each $a \in G$. By contrast, we call $A \in C(G)$ *layer-closed* if A is an intersection of minimal primes of G. With A layer-closed in G, let us denote $M(A)=\{P \in \mathbf{Min}(G)\colon P \not\supseteq A\}$ and $V(A)=\{P \in \mathbf{Min}(G)\colon P \supseteq A\}$. The following proposition collects some of the elementary facts about formation of closures and interiors in the hull-kernel topology on $\mathbf{Min}(G)$. Its proof is straightforward and will therefore be omitted.

2.1. PROPOSITION. *Suppose that $A \in C(G)$ is layer-closed. Then*:

i) *$M(A)$ is open in* $\mathbf{Min}(G)$, *and every open set is of this form.*

ii) *$V(A)$ is closed, and every closed set is this form.*

iii) *For each $a \in G$, $M(a)=M(a'')=V(a')$. Hence,* $\mathbf{Min}(G)$ *has a base of closed-open (clopen) sets, and is therefore totally disconnected.*

iv) *$cl(M(A))=V(A')$ and in $(V(A))=M(A')$. (in* (–) denotes the interior.)

v) *$M(A)$ is clopen $\leftrightarrow$ $A\boxplus A'$ is very large $\leftrightarrow$ $V(A)$ is clopen. If this is the case, then A is a polar.*

(Note: It should be obvious that every polar is layer-closed. If not so obvious, the reader might look at [BKW], Corollary 3.4.2.)

To determine under what conditions $\mathbf{Min}(G)$ is compact – and therefore a boolean space – let us recall another concept defined in [CM2]. We say that G is *complemented* if for each $x \in G$ there is a $y \in G$ so that $|x| \wedge |y|=0$ and $|x| \vee |y|$ is a unit. It should be clear that G is complemented if and only if $\mathbf{Pr}(G)$ is a *subalgebra* of $P(G)$.

If G is *laterally complete* – that is, if every set of pairwise disjoint elements has a supremum – then G is complemented. On the other hand, there are plenty of interesting examples of complemented ℓ-groups which are not laterally complete. For instance, if I is a set let B_I be the ℓ-group of all bounded real-valued functions defined on I. $C([0,1])$, the group of real-valued, continuous functions on the unit interval $[0,1]$, is also complemented. In general, the ℓ-group of all continuous, real-valued functions $C(X)$, defined on the topological space X is complemented if and only if for each cozero set V there is a cozero set W disjoint from V, so that $V \cup W$ is dense in X. (Recall: a *cozero set* is a set of the form $\mathrm{coz}(f)=\{x \in X\colon f(x) \neq 0\}$, for some continuous, real-valued function f.)

The following example was pointed out to us by Tony Hager. Let $\beta\mathbb{N}$ stand for the Stone-Čech compactification of the (discretely topologized) set of natural numbers, and $X=\beta\mathbb{N}\setminus\mathbb{N}$, with the (compact) subspace topology. From [GJ], chapter 6: X has no dense, proper cozero sets. However, it does have cozero sets which are not closed.

It was shown in [CM2] that if G is complemented, then $A \in C(G)$ is very large if and only if A contains a unit of G.

Now, the theorem we are about to state already appears in [LZ] as Theorem 37.4. for vector lattices. For the analogous result in f-rings, using annihilators, we refer the reader to Henriksen-Jerison [HJ].

2.2. THEOREM. *For any ℓ-group G,* $\mathbf{Min}(G)$ *is compact if and only if G is complemented.*

PROOF. (Necessity) Suppose that $0<a\in G$; then for each $P\in \mathbf{Min}(G)$ for which $a\in P$, select $b_P>0$ such that $b_P\in a'\setminus P$. Then the family

$$\{M(a)\} \sqcup \{M(b_P)\colon P\in \mathbf{Min}(G),\ a\in P\}$$

covers $\mathbf{Min}(G)$, and so, owing to the compactness, we conclude that the space is covered by a finite number of them. Furthermore, if $Q\in M(a)$ then $Q\notin M(b_P)$, and so $M(a)$ cannot be omitted. Thus

$$\mathbf{Min}(G) = M(a)\cup M(b_1)\cup \cdots \cup M(b_k),$$

where $b_1=b(P_1)$ $(1\le i\le k)$, and if we set $b=b_1\vee\cdots\vee b_k$, it is easily verified that b complements a.

(Sufficiency) By way of contradiction, suppose G is complemented but $\mathbf{Min}(G)$ has an open cover with no finite subcover. This means that for a suitable infinite ordinal number σ we may write

$$\mathbf{Min}(G) = M(a_1)\cup M(a_2)\cup \cdots \cup M(a_\tau)\cup \cdots,$$

over all ordinals $\tau<\sigma$, so that each $M(a_\tau)$ fails to be in the union of the preceding members of the cover; (which says that each $a_\tau\neq 0$, and so we may assume that each $a_\tau>0$.) The plan is to disjointify the cover, by transfinite induction.

Begin with a complement $b_1>0$ of a_1; (since a_1 is not a unit b_1 is, indeed, not zero, and so $M(b_1)\neq\emptyset$.) Then $M(b_1)$ is contained in the union of the $M(a_\tau)$, with $2\le\tau<\sigma$. Letting $b_\tau=b_1\ a_\tau$, we have that $M(b_1)=\bigsqcup\{M(b_\tau)\colon 2\le\tau<\sigma\}$. Now we replace the original cover of $\mathbf{Min}(G)$ by $\{M(a_1)\}\sqcup\{M(b_\tau)\colon 2\le\tau<\sigma\}$. It should be clear that $M(a_1)$ is disjoint from the others. We also retain the property that no $M(b_\tau)$, $(2\le\tau<\sigma)$ is in the union of the preceding members of the cover.

Suppose that for an ordinal β we have that a cover $\{M(c_\tau)\colon \tau<\sigma\}$ so that i) for each $\alpha<\beta$ and all $\delta<\sigma$, with $\alpha\neq\delta$, $c_\alpha\wedge c_\delta=0$, and ii) no $M(c_\tau)$ is contained in the union of the preceding members of the cover. Let $y>0$ be a complement of c_β. Then, as in the previous paragraph, and for $\tau>\beta$, replace each $M(c_\tau)$ by $M(y\wedge c_\tau)$. Then $\{M(c_\alpha)\colon \alpha\le\beta\}\sqcup\{M(yc_\tau)\colon \tau>\beta\}$ is once more a cover of $\mathbf{Min}(G)$ in which each $M(c_\alpha)$ $(\alpha\le\beta)$ is disjoint from every other member of the cover, and so that no member is contained in the union of the preceding ones.

By transfinite induction we obtain a cover of $\mathbf{Min}(G)$ $\{M(x_\tau)\colon \tau<\sigma\}$ for which $x_\mu\wedge x_\pi=0$, for all $\mu\neq\pi$, with each $x_\tau>0$. Now, for each $\tau<\sigma$, let $y_\tau>0$ be a complement of x_τ. Since no two of the y_τ are disjoint they generate, by finite intersection, a filter of positive elements of G, which is contained in an ultrafilter Y. By the Lemma on Ultrafilters $P=U\{x'\colon x\in Y\}$ is a minimal prime, which obviously contains every x_τ. This violates the premise that the $M(x_\tau)$ cover $\mathbf{Min}(G)$, and we've achieved the contradiction we were working for.

We therefore must conclude that if G is complemented then its space of minimal primes is compact. ■

To motivate the study of rigid subgroups, let us remind the reader of a concept introduced in [CM1]: suppose that G is an ℓ-subgroup of the ℓ-group H; we say that G is a *signature* for H if the map $P \to P \cap G$ is a boolean isomorphism from $P(H)$ onto $P(G)$. This is one of the many types of extensions considered in the theory of ℓ-groups, which are defined by some bijective condition on the contraction map $C \to C \cap G$, (and where C ranges over a suitable family of convex ℓ-subgroups of H. For the more expert reader we cite as examples the a- and a^*-extensions, although we have no need of these concepts here.)

Summarizing then, if G is an ℓ-subgroup of H, we call G a *signature* for H - roughly speaking - when $P(G)=P(H)$. Now we ask, what type of extension will leave $\mathbf{Min}(G)$ homeomorphic to $\mathbf{Min}(H)$?

We say that G is *rigid* in H - or that H is a *rigid extension* of G - if for each $h \in H$ there is a $g \in G$ so that $h''=g''$. (Note: in these situations, where we compare polarities of two groups, we shall typically use $(-)'$ for polars in the larger group and $(-)^{\perp}$ in the smaller.) It does turn out, as we shall presently demonstrate, that if G is rigid in H then $\mathbf{Min}(G)=\mathbf{Min}(H)$, but not the reverse. We will show that if both groups are complemented then the converse holds. But, for the general case, we need the following weaker (as well as less appealing) notion.

Continuing with G as an ℓ-subgroup of H, we call G an *r-subgroup* of H - or H and *r-extension* of G - if for each $0<h \in H$ and each $P \in \mathbf{Min}(H)$ which does not contain h, there exists a $g(P)$ in G but outside P, so that $g(P)'' \subseteq h''$. It should be obvious that a rigid subgroup is an r-subgroup as well.

The following example shows that an r-subgroup need not be rigid. Let $H=\{(s,n)$: s is a sequence of integers which is eventually constant; $n \in Z\}$, and order H by $(s,n)>0$ if each $s_m \geq 0$ and s is eventually a positive number, while if s is eventually 0 then $n \geq 0$. Let $G=\{(s,n) \in H$: s is eventually $0\}$; then G is an r-subgroup, but it is not rigid. Indeed, it is easy to verify that if $G \in C(H)$ - as is the case in this example - then G is rigid in H if and only if it is very large in H. (The reason is this: if $G \in C(H)$, then the contraction $P \to P \cap G$ is one-to-one on the primes if H which fail to contain G.)

2.3. PROPOSITION. *Suppose that G is an ℓ-subgroup of H. Then*

i) *if H is complemented and G is an r-subgroup of H, then G is rigid in H, and hence (by* ii) *below,) G is complemented as well.*

ii) *G is an r-subgroup of H if and only if the map $P \to P \cap G$ is a homeomorphism of* $\mathbf{Min}(H)$ *onto* $\mathbf{Min}(G)$.

PROOF. i) Suppose that $0<h \in H$. For each minimal prime P of H which does not contain h. Select $g(P)>0$ such that $g(P) \in G \setminus P$ and g'' is contained in h''. By Theorem 2.2., $\mathbf{Min}(H)$ is compact and, therefore, the closed set $M(h)= =M(h,H)$ is, likewise, compact. Now, what we have is a cover $\{M(g(P))= =M(g(P),H)$: $h \notin P\}$ for $M(h)$. Consequently, there exist $g_1, g_2, \ldots, g_k$ ($g_i= =g(P_i)$) so that $M(h)=M(g_1) \cup \cdots \cup M(g_k)$. If we let $g=g_1 \vee \cdots \vee g_k$, then $M(h)=M(g)$, which means that $h''=g''$.

ii) ($\to$) The most remarkable point is that if G is an r-subgroup of H, then

$P\cap G\in\mathbf{Min}(G)$ whenever $P\in\mathbf{Min}(H)$. The other attributes of the contraction map are straightforward. Observe at the outset, that if $P\in\mathbf{Min}(H)$ and $P\not\supseteq G$, then $P\cap G$ is prime. In addition, the definition of r-subgroup guarantees that $P\not\supseteq G$, for every minimal prime P of H. What must be shown is that $P\cap G$ is minimal; however, we must take care of other matters first.

If P_1 and P_2 are distinct minimal primes of H there must be an $h>0$ such that $h\in P_1\setminus P_2$. Then we can find $0<a\in G\setminus P_2$, and so that $a''\subseteq h''$. But then $a\in P_1$, and therefore $P_1\cap G$ and $P_2\cap G$ are also distinct. Thus the contraction map is one-to-one on $\mathbf{Min}(H)$. On the other hand, suppose $N\in\mathbf{Min}(G)$; then $F(N)=\{0<g\in G\colon g\notin N\}$ is a filter (of H), and is therefore contained in an ultrafilter V of H. Letting $P=\bigcup\{x'\colon x\in V\}$, we obtain a minimal prime of H which contracts to N, proving that the contraction map is surjective.

Now, if $P\in\mathbf{Min}(H)$ then $P\cap G$ is prime in G and, therefore, contains a minimal prime Q of G. By the preceding paragraph, $Q=P_0\cap G$ for a suitable minimal prime P_0 of H. But, as $P_0\cap G\subseteq P\cap G$, we also conclude from the same paragraph that $P_0\subseteq P$, and hence $P_0=P$. This at last gives that $P\cap G\in$ $\in\mathbf{Min}(G)$.

The above arguments prove that $P\to P\cap G$ is a bijection between $\mathbf{Min}(H)$ and $\mathbf{Min}(G)$. The assumption of r-containment is designed to make the map open. It is easy to show that it is continuous.

($\leftarrow$) We assume that the contraction map is a homeomorphism from $\mathbf{Min}(H)$ onto $\mathbf{Min}(G)$. Then for each $h>0$ in H, the image of $M(h,H)$ is open. By the definition of the hull-kernel topology, this means that for each $P\cap G$ for which $h\notin P$, there is a basic open set $M(g(P),G)$ contained in the image of $M(h,H)$, such that $P\in M(g(P),G)$; this translates into: $g(P)\notin P$ and $g(P)''\subseteq h''$, proving that G is an r-subgroup of H. ■

This is as good a place as any to observe that an r-subgroup is also (evidently so) a signature. We point out as well that a rigid subgroup is a signature for which the contraction $P\to P\cap G$ is a lattice isomorphism from $\mathbf{Pr}(H)$ onto $\mathbf{Pr}(G)$.

Rigid subgroups may be viewed from another point of view, by considering a different contraction map. First, let us recall a notion introduced by Bigard: $A\in C(G)$ is called a *z-subgroup* if $a\in A$ implies that $a''\subseteq A$.

These subgroups occur in the literature under different names. S.J. Bernau calls them z-ideals in his paper "Topologies on structure spaces of lattice groups" in Pacific J. Math. 42 (1972), 557–568. A.S. Bodarev calls them pseudo-normal ideals in his paper "The presence of projections in quotient lineals of vector lattices" in Dokl. Akad. Nauk UzSSR 8 (1974), 5–7. C.B. Huijsmans and B. de Pagter [HudP1], [HudP2], [dP], [HudP3] call them d-ideals in the case of vector lattices (Riesz spaces). Some of our upcoming results are closely related to work of the latter authors.

It is well known - see [BKW] - that every minimal prime and every polar are z-subgroups; consequently, every layer-closed subgroup is a z-subgroup. The converse is false, in general; we shall give an example a bit later on, when we also take up conditions for the converse to be satisfied.

We wish to point out the following as well: if A is a z-subgroup then $A = \bigcup\{g'': g \in A\}$. The z-subgroups form a lattice under inclusion, and it is a complete lattice, which we shall denote by $z(G)$. In this lattice all infima agree those in $C(G)$, but not the suprema.

We have arrived at the following characterization of rigidity:

2.4. PROPOSITION. *Suppose G is an ℓ-subgroup of H. Then G is rigid in H precisely when the contraction map $B \to B \cap G$ is a lattice isomorphism from $z(G)$ onto $z(G)$.*

PROOF. ($\leftarrow$) We begin with the observation that, for each $g \in G$, g'' is the smallest z-subgroup of G containing g. Indeed, in $z(G)$, the principal polars are precisely the *join inaccessible* elements; meaning that if $g'' = U\{A_i: i \in I\}$ and the A_i are upward directed, then $g'' = A_j$, for a suitable $j \in I$. The lattice isomorphism contracting z-subgroups of H of G must then send principal polars to principal polars, proving that G is rigid in H.

($\rightarrow$) For each $A \in z(G)$, let A^z denote the z-subgroup of H generated by A. It should be obvious that $A^z = U\{a'': a \in A\}$. Now, since $g'' \cap G = g\perp\perp$, for each $g \in G$, it follows that $A^z \cap G = A$. This shows that the map $B \to A \cap G$ is surjective. (The reader should verify that, owing to the rigidity, $B \in z(H)$ implies that $B \cap G \in z(G)$.)

Finally, suppose that B_1, $B_2 \in z(H)$, with $B_1 \not\subseteq B_2$, and pick $0 < b \in B_1 \setminus B_2$. Find an $a \in G$ for which $a'' = b''$; then $a \in B_1 \cap G \setminus B_2$. This establishes that the contraction map is one-to-one. As it clearly preserves inclusion, and is therefore a lattice isomorphism, the proof is complete. ∎

2.4.1. COROLLARY. *If G_1 is rigid in G_2 and G_2 is rigid in G_3 then G_1 is rigid in G_3. The property of being an r-subgroup is, likewise, transitive.*

PROOF. The first statement follows immediately from Proposition 2.4. The second is a consequence of Proposition 2.3.ii). ∎

We now turn to some more specialized remarks, and also to a few examples.

2.5.1. Recall that an ℓ-subgroup G of H is said to be an *a-subgroup* if the contraction map $C \to C \cap G$ is a lattice isomorphism from $C(H)$ onto $C(G)$. In terms of the kinds of extensions one might consider, which are defined by placing a demand on the contraction map, this one is the strongest. The reader should observe that if H is hyper-archimedean, then by 1.2a), G is an r-subgroup of H if and only if it is an a-subgroup. And with regard to Proposition 2.4, note that in this context every convex ℓ-subgroup is a z-subgroup.

2.5.2. Let I be an infinite set and $H = Z^I$, with pointwise operations, while G is the ℓ-subgroup of finitely nonzero functions. Then G is large – and, indeed, dense - in H; it is therefore a signature, but not an r-subgroup. (Recall that G is *large* in H if $C \cap G \neq 0$ whenever $C \neq 0$ in $C(H)$; G is *dense* in H if for each $0 < h \in H$ there is a $g \in G$ so that $0 < g \leq h$. Clearly every dense subgroup is large.)

2.5.3. Let H be the example following Theorem 2.2., while $G = \{(s, 0)$: s is eventually constant$\}$. Then $\mathbf{Min}(G)$ is compact, whereas $\mathbf{Min}(H)$ is not: its point at infinity is isolated! The contraction map $P \to P \cap G$ is a continuous bijection which is not open.

2.5.4. For this item let us restrict our attention to archimedean ℓ-groups. For any r-extension H of G, G is large in H - see Theorem 11.1.15. in [BKW] - and it follows that H is contained in the essential closure $E(G)$ of G. (For an account of the essential closure of an archimedean ℓ-group, see [C2].) Next, we introduce a partial ordering on the r-extensions of G by: $H_1 \leq_r H_2$ is an r-extension of H_1. It is then straightforward to verify that the set of all r-extensions of G is inductive. Therefore, G has an *r-closure* in $E(G)$. What can we say about such an r-closure? Is it unique up to ℓ-isomorphism? Or is the best that we can expect that two r-closures of G have homeomorphic spaces of minimal primes?

Let us, however, discourage even the bold reader: suppose that G stands for the ℓ-groups of finitely nonzero real sequences (with pointwise operations). Then $E(G) = R^N$, that is, the group of all real sequences. As we shall see momentarily, each r-extension of G must have a very large basis - a concept we shall recall in the next paragraph. We then appeal to [CM2] to conclude that G is r-closed. So an r-closed ℓ-group need not be complemented.

In an ℓ-group G - no longer necessarily archimedean - the element $b > 0$ is said to be *basic* if the set $\{x \in G$: $0 \leq x \leq b\}$ is a chain. It is well known that b is basic if and only if $G(b)$ is an o-group; (see [BKW], section 7.3.) G is said to *have a basis* if it possesses a maximal pairwise disjoint set $\{b_1$: $i \in I\}$ consisting of basic elements. G is said to have a *very large basis* if it has a basis $\{b_1$: $i \in I\}$ so that the cardinal sum $\boxplus\{b_1''$: $i \in I\}$ is a very large subgroup.

The properties of having a basis or a very large basis are carried over to an r-extension because of the following lemma:

2.6. LEMMA. *Suppose that G is an ℓ-group and $P \in \mathbf{Min}(G)$. Then the following are equivalent*:

(a) *P is isolated in* $\mathbf{Min}(G)$;

(b) *P is a polar.*

PROOF. Straightforward. ■

Now, using Proposition 1.10. in [CM2], which states that G has a very large basis if and only if every minimal prime of G is a polar, we obtain:

2.6.1. COROLLARY. *Suppose that G is an r-subgroup of H. Then*:

i) *G has a basis if and only if H has a basis, which, in turn is the case precisely when* $\mathbf{Min}(G)$ *has a dense, discrete subspace.*

ii) *G has a very large basis if and only if H does, which occurs just when* **Min**(G) *is discrete.*

PROOF. Apply Proposition 2.3., Lemma 2.6. and observe that G has a basis precisely when the polar primes have trivial intersection. ■

2.6.2. By contrast, the property of being finite-valued is not preserved under a rigid extension. (The reader might consult [BKW], section 6.4., for the topic of finite-valued ℓ-groups, or else skip this example entirely. We shall deal with rigid extensions of finite-valued ℓ-groups in an upcoming article.)

Let us briefly recall the Hahn-group construction: if Γ is any root system, and for each $\alpha \in \Gamma$, R_α denotes a subgroup of R, we let $V(\Gamma, R_\alpha)$ stand for the group of functions f: $\Gamma \to \bigsqcup \{R_\alpha: \alpha \in \Gamma\}$ with $f(\alpha) \in R_\alpha$ for each $a \in \Gamma$, and so that $\mathrm{coz}(f) = \{\alpha \in \Gamma: f(\alpha) \neq 0\}$ has no infinite ascending sequences. It is well known – see [CHH] – that $V(\Gamma, R_\alpha)$ becomes an ℓ-group when one defines $f > 0$ provided $f(\alpha) > 0$ for each maximal α in $\mathrm{coz}(f)$.

Now consider the root system $\Gamma = \{\alpha_n: n \in \mathbb{N}\} \cup \{\beta_n: n \in \mathbb{N}\}$, with $\alpha_n < \beta_n$ and $\beta_1 > \beta_2 > \cdots$. Let $G = \{f \in V(\Gamma, Z): f$ is finitely nonzero$\}$ and H be the ℓ-subgroup generated by G and u, the element defined by $u(\beta_n) = 0$, for each $n \in \mathbb{N}$, while $u(\alpha_n) = 1$, for each $n \in \mathbb{N}$. Then G is rigid in H and G is finite-valued, whereas H is not.

We close this section with a discussion turning on the following question: when is an ℓ-group rigid in its lateral completion? Our answer is Theorem 2.7., but let us first remind the reader of what a lateral completion is. For further illumination we refer to [C1] and [Be]. In any event, every ℓ-group G possesses a *lateral completion* H: H is laterally complete, contains G as a dense ℓ-subgroup, and no proper ℓ-subgroup of H contains G and is laterally complete.) We recall that the lateral completion G^L of G is unique up to ℓ-isomorphism, and that G^L can be constructed from G transfinitely by adjoining suprema of pairwise disjoint sets.

Before stating Theorem 2.7 we need to review a few more items. First, recall that a topological space X is called a *Stone space* if it is compact, Hausdorff and extremally disconnected – which means that the closure of every open set is open. Stone spaces are precisely the duals of complete boolean algebras. Secondly, the remark following Theorem 2.9. in [ACM] implies that **Pr**(G) is a complete sublattice of $P(G)$ if and only if **Pr**$(G) = P(G)$.

Finally, note that complemented ℓ-groups are what Huijsmans and de Pagter call *d-regular*; (more on this term later.) Furthermore, in the context of vector lattices their Proposition 4.10. of [HudP3] already gives the equivalence of (b) and (c) below.

The proof of the equivalences of (b) through (e) is sufficiently like that of the corresponding lattice-theoretic result in [Sp], that we shall omit it.

2.7. THEOREM. *For a complemented ℓ-group G the following are equivalent:*

(a) *G is rigid in G^L.*
(b) **Min**(G) *is a Stone space.*
(c) **Pr**(G) *is a complete boolean algebra.*
(d) **Pr**(G) *is a complete sublattice of* $P(G)$.
(e) **Pr**$(G)=P(G)$.

PROOF. Since for a laterally complete ℓ-group (e) is obviously satisfied, we conclude that **Min**(G^L) is a Stone space. By Proposition 2.3 it then follows that (a) implies (b). Conversely, if (b) holds, then, since G is certainly a signature for G^L, the contraction map $P \to P \cap G$ is a boolean isomorphism from $P(G^L) = \mathbf{Pr}(G^L)$ onto $P(G) = \mathbf{Pr}(G)$. By an earlier remark G must be rigid in G^L. ■

3. THE CONDITION ZLC

Upon introduction of z-subgroups we remarked that every layer-closed subgroup had to be a z-subgroup, but that the converse was false. Here's an example: let $\Gamma = \{\alpha_0, \alpha_1, \ldots\}$ ordered by $\alpha_0 > \alpha_n$, for each $n \geq 1$, and form $G = \{f \in V(\Gamma, Z)$: f is finitely nonzero$\}$. Then $A = \{g \in G$: $g(\alpha_0) = 0\}$ is a z-subgroup as well as a nonminimal prime.

We shall employ the abbreviation ZLC for the condition: every proper z-subgroup is layer-closed. Every complemented ℓ-group satisfies ZLC and more: let us call an ℓ-group *locally complemented* if $G(x)$ is complemented for each $x \in G$. And observe immediately that since each $G(x)$ is very large in x'', then G is locally complemented if and only if each x'' is complemented. In addition, note that if G is locally complemented then **Min**(G) is locally compact. (The converse is false: let G be the example cited in the preceding paragraph. G has a strong unit, yet is not complemented; hence it is not locally complemented either. However, **Min**(G) = **Min**(A), and homeomorphic to the discrete space $\mathbb{N}$, which is locally compact.

The term "locally complemented" is synonymous with "d-regularity" in the work of Huijsmans and de Pagter. They show - albeit for vector lattices - that d-regularity is equivalent to the assumption that every maximal z-subgroup is a minimal prime; see Theorem 9.5. in [HudP2]. This is a consequence of the next result, the main theorem of this section. It gives somewhat more information about the role of minimal primes in locally complemented ℓ-groups.

3.1. THEOREM. *An ℓ-group G satisfies ZLC if and only if it is locally complemented.*

We prove Theorem 3.1 in stages; here is the first lemma:

3.2. LEMMA. *If the ZLC holds in G then it does as well in each x'', for each $x \in G$.*

PROOF. Suppose that A is a proper z-subgroup of x'', and that G satisfies the ZLC. Let B be the largest convex ℓ-subgroup of G for which $B \cap x'' = A$;

observe that $0<b\in B$ precisely when $b\wedge x\in A$. (We may – and do – without loss of generality take $x>0$.) Then B is a proper z-subgroup of G: to see this, notice first that since $A\neq x''$ it follows that $B\neq G$; and if $0\leq y\in b''$, with $0\leq b\in B$, then $y\wedge x\in(b\wedge x)''\subseteq A$, which implies that $y\in B$. This means that B is an intersection of minimal primes of G, and by contraction to x'', we see that A is layer-closed in x''. ■

Thus, to prove the necessity in Theorem 3.1 what needs to be done is to establish:

3.3. LEMMA. *If G has a unit and satisfies the ZLC then G is complemented.*

PROOF. We begin by showing that if $A\in C(G)$ is very large then A contains a unit of G. Because of the ZLC, A^z, the z-closure of A in G, is none other than G. Thus if u is a unit of G then $u\in a''$, for some $a\in A$, which makes a a unit of G.

To finish this proof we prove that **Min**(G) is compact. Suppose that $\{A_1\colon i\in I\}$ is a family of z-subgroups for which the open sets $M(A_1)$ cover **Min**(G). Then $V\{A_1\colon i\in I\}$, taken in $C(G)$, is very large in G, and by the comments in the first paragraph, this supremum contains a unit $u>0$ of G. But then $u\in A_1\vee A_2\vee\cdots\vee A_n$, for a suitable finite collection of indices $1,2,\ldots,n$ in I; (remember that $C(G)$ is an algebraic lattice.) In turn, this means that $A_1\vee A_2\vee\vee\cdots\vee A_n$ is very large in G, and hence that the set $\{M(A_1)\colon i=1,2,\ldots,n\}$ covers **Min**(G). ■

For the converse of Theorem 3.1, we require, once again, two lemmas. The first allows us to pass with ZLC from the local to the global.

3.4. LEMMA. *Suppose that each G(x) satisfies the ZLC; then so does G.*

PROOF. Start with a proper z-subgroup A of G; pick $0<x\in G\setminus A$. Then $A\cap G(x)$ is a proper z-subgroup of $G(x)$, whence it is an intersection of minimal primes of $G(x)$. Any one of these primes is of the form $P\cap G(x)$, where $P\in\mathbf{Min}(G)$, $x\notin P$ and $P\supseteq A$. This suffices to prove that A is layer-closed. ■

And now, armed with Lemma 3.4, all we have to do is prove that the sufficiency in Theorem 3.1. holds globally.

3.5. LEMMA. *If G is complemented then the ZLC is satisfied.*

PROOF. Suppose that A is a proper z-subgroup of G. The first step is to show that A is contained in *some* minimal prime. Now $A=\cup\{x_i''\colon \text{each } x_i>0\}$, where the x_1 form an upward directed subset of A. Pick a complement $y_i>0$ of x_i. (Since A is proper none of the x_i are units, and therefore none of the complements are zero.) The y_i then lie in an ultrafilter V of G. Letting $P=\cup\{y'\colon y\in V\}$, we have a minimal prime of G which contains A.

Next, if $0<x\notin A$ then $A\cap G(x)$ is a proper z-subgroup of $G(x)$. On the other hand, since $\mathbf{Min}(G(x))$ is homeomorphic to $M(x)$, a closed, and therefore, compact subspace of $\mathbf{Min}(G)$, Theorem 2.2 gives us that $G(x)$ is complemented. We conclude that $A\cap G(x)$ is contained in a minimal prime Q_0 of $G(x)$. But, as before, $Q_0=Q\cap G(x)$, for some $Q\in\mathbf{Min}(G)$, with $x\notin Q$ and $Q\supseteq A$. This proves that A is layer-closed. ■

The proof of Theorem 3.1. is now complete.

The next observation is a special case of the theorem, but it merits special mention. See Theorem 9.8(i) in [HudP2] for comparison.

3.6. PROPOSITION. *For an ℓ-group G with unit, the following are equivalent:*

a) *G is complemented.*
b) *G is locally complemented.*
c) *G satisfies the ZLC.*

In conclusion of this section let us mention a condition which, at first glance, appears to be stronger than ZLC but, as Proposition 3.7. will demonstrate, is not.

Let us call a convex ℓ-subgroup A of G *strongly layer-closed* is every prime P of G which covers A is minimal. Obviously, any strongly layer-closed subgroup is layer-closed. We shall employ ZSLC for the condition: every proper z-subgroup is strongly layer-closed.

3.7. PROPOSITION. *An ℓ-group G is locally complemented if and only if the ZSLC holds in G.*

PROOF. Only the necessity is at issue here, and, as with our lemmas in this section, it suffices to prove the claim globally. We therefore assume that G is complemented.

Now, suppose that A is a proper z-subgroup of G, and P is a prime that covers A; we must show that $P\in\mathbf{Min}(G)$. Note that it is sufficient to prove that P itself is a z-subgroup. By a variation on the Lemma on Minimal Primes, using filters which do not intersect A, one can prove that if $0<a\in P\setminus A$ then there is an element $b>0$, so that $b\notin P$, yet $a\wedge b\in A$. Now, suppose that $0<a\in P$; we shall demonstrate that $a''\subseteq P$. Without loss of generality, assume that $a\notin A$; pick $b>0$ as indicated before. Then, since A is a z-subgroup, $a''\cap b''\subseteq A\subseteq P$, and as P is prime and $b''\not\subseteq P$, it follows that $a''\subseteq P$. ■

(Note: Originally we had proved this proposition for representable ℓ-groups. Recently Rick Ball pointed out to us that such an assumption was unnecessary.)

We point out that our proof employs properties of the (Brouwerian) lattice $C(G)$, whereas Huijsmans and de Pagter rely on properties of vector lattices.

4. THE BOOLEAN ALGEBRA A $P_c(G)$

This section is dedicated to one construction; we identify a "canonical" compactification of $\mathbf{Min}(G)$. Some applications to projectable ℓ-groups are also considered.

Before getting underway, let us adopt the notation $B^\wedge$ for the dual space of a boolean algebra B.

Suppose that G is an ℓ-group, and let $P_c(G)$ denote the set of polars P for which $P \boxplus P'$ is very large in G. (Recall that these are precisely the layer-closed subgroups that determine clopen subsets $M(P)$ of $\mathbf{Min}(G)$.) It should be obvious that $P_c(G)$ is a subalgebra of $P(G)$ containing $\mathbf{Pr}(G)$. Since every polar is the supremum of principal ones, it follows that $P_c(G)$ is a complete sublattice of $P(G)$ if and only if $P(G) = P_c(G)$. Notice as well that in a laterally complete ℓ-group G, $P(G) = P_c(G) = \mathbf{Pr}(G)$. We observe, finally, leaving the details to the reader, that G is complemented if and only if $P_c(G) = \mathbf{Pr}(G)$.

What we want to do is show that $P_c(G)^\wedge$ is a compactification of $\mathbf{Min}(G)$, in such a way as to make it clear that G is complemented precisely when $\mathbf{Min}(G)$ is homeomorphic to $P_c(G)^\wedge$.

For each minimal prime P of G define a character k_P on $P_c(G)$ by:

$$k_P(Q) = \begin{cases} 1 & \text{if } Q \not\subseteq P, \\ 0 & \text{if } Q \subseteq P. \end{cases}$$

In the following proposition we record the fundamental properties of the map k.

4.1. PROPOSITION. *k: $\mathbf{Min}(G) \to P_c(G)^\wedge$ is a topological embedding, the image of which is dense in $P_c(G)^\wedge$. G is complemented if and only if* $\mathbf{Min}(G) = P_c(G)^\wedge = \mathbf{Pr}(G)^\wedge$.

SKETCH OF PROOF. Let us verify that k is one-to-one, open, and that $k(\mathbf{Min}(G))$ is dense in $P_c(G)^\wedge$; we shall leave the rest to the reader, including the (straightforward) verification that each k_P is a boolean character.

Suppose $P \neq N$ in $\mathbf{Min}(G)$, and $a \in P \setminus N$. Then $a'' \subseteq P$, but $a'' \not\subseteq N$; since $a'' \in P_c(G)$, we have that $k_P(a'') = 0$ while $k_N(a'') = 1$, proving that k is one-to-one. Next, for each $g \in G$, $k_{(M(g))} = \{k_P \colon P \in \mathbf{Min}(G), g \notin P\} = \{k_P \colon k_P(g'') = 1\}$, which is a relatively open set. Thus, k is an open mapping.

As for the density of the image of $\mathbf{Min}(G)$ in $P_c(G)^\wedge$, note that the basic open sets of $P_c(G)^\wedge$ are the ones of the form

$$M(a) = \{\phi \in P_c(G)^\wedge \colon \phi(a'') = 1\},$$

with a ranging over the elements of G. If $a'' = G$ then $M(a) = P_c(G)$ and there is nothing to prove. If $a' \neq 0$ there is a minimal prime P such that $P \supseteq a'$ and $k_P(a'') = 1$, which means that $M(a)$ and the image of k intersect nontrivially, proving that the map embeds $\mathbf{Min}(G)$ densely in $P_c(G)^\wedge$. ∎

In an upcoming article [CM3], on adjunction of units to ℓ-groups, this embedding k will play a significant role. We also should point out that $P_c(G)$ turns out to be the largest zero-dimensional compactification of $\mathbf{Min}(G)$; (for details on this subject the reader is referred to [PW].)
For the rest of this article we shall study $P_c(G)$ in certain specialized situations.

Recall that an ℓ-group G is said to have *stranded primes* if every prime of G contains a unique minimal prime. (In [BKW] such ℓ-groups are called *semi-projectable*; Huijsmans and De Pagter refer to them as *normal*.) Dedekind-MacNeille complete ℓ-groups have long been known to posses this property; (see 11.2.3 in [BKW].) Indeed, they are special cases of *projectable ℓ-groups* – ℓ-groups G for which $G = x'' \boxplus x'$ for each $x \in G$ – and every projectable ℓ-group has stranded primes. The converse fails; consider, for instance, the ℓ-subgroup of integer-valued sequences generated by the finitely nonzero sequences, $(1, 0, 1, 0, \ldots)$ and $(1, 2, 3, 4, \ldots)$.

4.2. PROPOSITION. *If G is projectable then $P_c(G)$ is the subalgebra of cardinal summands of G.*

PROOF. Observe at the outset that a projectable ℓ-group is complemented if and only if it has unit. Hence every projectable ℓ-group is locally complemented, and, therefore, by Theorem 3.1, satisfies the ZLC. So if we could show that, for each $Q \in P_c(G)$, $Q \boxplus Q'$ is a z-subgroup, it would follow that Q is a cardinal summand. As the lemma to follow establishes, this precisely what happens.

We make note first of Theorem 4.4. from [HudP1], which gives us half the lemma; (for vector lattices, but the proof carries over to our situation.) The novelty here is that the strandedness of primes is equivalent to $z(G)$ being a sublattice of $C(G)$.

4.3. LEMMA. *Suppose that G is a complemented ℓ-group. Then G has stranded primes precisely when $z(G)$ is a sublattice of $C(G)$.*

PROOF. (Sufficiency) Suppose that P_1 and P_2 are distinct minimal primes. Because of the ZLC, their supremum as z-subgroups is G, and thus also their supremum in $C(G)$. But then no proper prime of G can contain them both. ■

Now, let us complete the proof of Proposition 4.2.: if A and B are z-subgroups of G, with G projectable, then, for each x in G $x'' \cap (A \vee B)$ is a proper z-subgroup of x'', by Lemma 4.3. Since x'' satisfies the ZLC, $x'' \cap (A \vee B)$ is an intersection of minimal primes of x''; this is enough to show that there is a minimal prime P of G containing $A \vee B$ so that $x \notin P$. Hence $A \vee B$ is layer-closed and hence a z-subgroup.

Finally, pick $Q \in P_c(G)$ and apply the above to $Q = A$ and $Q' = B$. ■

To conclude, we state a result the proof of which uses Lemma 4.3 and the proof just presented. Compare with Theorem 9.2 of [HudP2]; the novelty, apart from the greater generality, is the sublattice condition.

4.4. PROPOSITION. *Suppose that G is locally complemented. Then G has stranded primes if and only if G is projectable if and only if $z(G)$ is a sublattice of $C(G)$.*

PROOF. If G has stranded primes and $x \in G$ then $G(x)$ is complemented and, by Lemma 4.3., $z(G(x))$ is a sublattice of $C(G(x))$. If $y \in G(x)$ then $y^{\perp\perp} \boxplus y^{\perp}$ - polars taken in $G(x)$ - is a z-subgroup of $G(x)$ and very large in it. Thus, $G(x) = y^{\perp\perp} \boxplus y^{\perp}$, proving that $G(x)$ is projectable, for every $x \in G$. This makes G itself projectable. Since the reverse implication is well known this suffices to establish the first equivalence.

As to the second, the proof of Proposition 4.2. shows that if G is projectable then $z(G)$ is a sublattice of $C(G)$. Conversely, observe that the first part of the proof of Lemma 4.3. only uses the ZLC. Since G is assumed to be locally complemented the ZLC is satisfied by Theorem 3.1, and so G has stranded primes. ■

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